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1991 J. Phys. A: Math. Gen. 24 1903

(http://iopscience.iop.org/0305-4470/24/8/029)

J. Phys. A: Math. Gen. 24 (1991) 1903–1914. Printed in the UK

An exactly solvable self-avoiding walks model: II. Triangular and honeycomb lattices

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Received 25 July 1990, in final form 8 January 1991

Abstract. A modified self-avoiding walks model previously proposed for the square lattice is applied to the triangular and honeycomb lattices. In the model, the walker is restricted not to take any turn which will put the walker in a direction rotated by more than a certain angle, Φ_{max}, from any of the directions previously taken. The generating functions are obtained, and various quantities are evaluated exactly. For all three (square, triangular and honeycomb) lattices, it is found that the model exhibits the characteristics reminiscent of one-dimensional self-avoiding walks in all directions, while strongly retaining the anisotropic effect of the direction of the first step.

For the honeycomb lattice, we have studied the model for $\Phi_{max} = \pi$ (A-model) and $\Phi_{max} = \frac{3}{2}\pi$ (B-model). Unlike other walks, the B-model exhibits a special property: the oscillations between even and odd steps in various quantities such as the number of N-step walks do not decay as the number of steps increases.

1. Introduction

Faced with insurmountable difficulties in the exact analysis of self-avoiding walks (SAWs), modified SAW models with additional restrictions have been actively investigated. Recently, we have also introduced one such model ([1], hereafter called I) for two-dimensional walks. In the model, the additional restriction is that all turns which will put the walker in a direction rotated by more than a certain angle, Φ_{max}, measured from *any* of the directions taken previously, are prohibited. The model is amenable to exact analysis and yet the results are not trivial. The model has been applied to the square lattice and the results were reported in I. To understand the nature of the walks such as critical exponents, universality, etc, additional results for other lattice structures are desired. For these purposes, we have studied the model for the triangular and honeycomb lattices, and we report the results in this paper.

The model is much more general than that of directed self-avoiding walks (DSAWs) [2–5] in the sense that it generates many more walks in addition to all of the walks permitted in DSAWs. Unlike in DSAW, the walker, in this model, may reach all of the quarters of the plane when Φ_{max} is chosen properly. The DSAW exhibits mixed behaviour, displaying the characteristics reminiscent of one-dimensional SAWs in certain directions

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0305-4470/91/081903+12$03.50

and random-walk (RW) like characteristics in other directions [2–5]. This model, however, shows the characteristics reminiscent to one-dimensional SAWs in *all* directions while strongly retaining the anisotropic effect of the direction of the first step.

The spiral self-avoiding walks (SSAWs) model exhibits different critical behaviour in the square and triangular lattices [6–11]. This model, however, shows the *same* critical behaviour in all three (square, triangular and honeycomb) lattices.

Due to the *wavy* nature of the walks in the honeycomb lattice, more than one value of Φ_{max} may be defined. We have studied the model for $\Phi_{max} = \pi$ (A-model) and $\Phi_{max} = \frac{3}{2}\pi$ (B-model). The noted property of the ordinary SAW on honeycomb lattice [12–17], that various quantities such as number of N-step walks exhibit strong oscillations between even and odd steps, is further strengthened in the B-model to such an extent that the oscillation becomes permanent. It is caused by the fact that the additional restriction of the model enhances the already favoured odd-step walks of the ordinary SAW. On the other hand, in the A-model, the additional restriction of the model suppresses the odd-step walks of the ordinary SAW, and the even–odd oscillation decays faster than in the ordinary SAW. Even in the B-model, however, the even–odd oscillation of the mean square end-to-end distance decays as $N \to \infty$.

For the sake of clarity and conciseness, we list the bulk of the results in the appendix in such a format that all relevant informations can be extracted easily. In section 2, we introduce the model and derive the generating functions for the triangular and honeycomb lattices. In section 3, we present a brief discussion of the results and compare them with other results.

2. The models and the generating functions

We fix the direction of the first step in a particular direction. The walks started out in other directions can be obtained easily by the symmetry considerations. In many occasions, such as in studying the persistency of the first step [18–20], the results containing only the walks started out in one particular direction are much more useful. Unless specified otherwise, all the results shown in this paper are for the walks started out in the particular direction defined.

For the studies of the walks in the triangular and honeycomb lattices, we need to define $u_\pm$ and $v_\pm$ directions as in figure 1, in addition to the $x_\pm$ directions in the Cartesian coordinate system.

In addition to the usual self-avoiding condition, we impose in this model a global restriction that the walker may not take any turn which will put the walker in a direction rotated by more than Φ_{max} from any of the directions taken previously. In general, Φ_{max} is defined as the angle that the walker will, when rotated by more than Φ_{max} from the direction previously taken, sight the line of the extension of the direction previously taken. When a lattice such as honeycomb has *wavy* lines of extension, more than one value of Φ_{max} may be defined. We report here the results for two values of Φ_{max} in the honeycomb lattice.

2.1. *Triangular lattice*

For the walks in the triangular lattice, we need to define the following basic generating

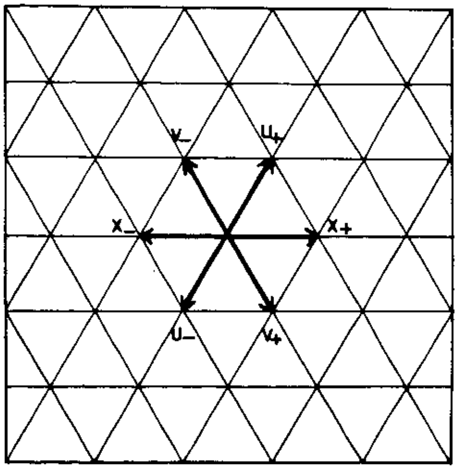

Figure 1. The directions of the walks in the triangular and honeycomb lattices.

functions:

$$G_1(x_+)=\frac{1}{1-x_+} \tag{1a}$$

$$G_1(x_+,x_-)=\frac{1-x_+x_-}{(1-x_+)(1-x_-)} \tag{1b}$$

$$G_3(x_+|u_+,v_-)=\frac{G_1(x_+)}{1-(u_++v_-)G_1(x_+)} \tag{1c}$$

$$G_4(x_+,x_-|u_+,v_-)=\frac{G_1(x_+,x_-)}{1-(u_++v_-)G_1(x_+,x_-)}. \tag{1d}$$

Here, $G_1(x_+)$ is the generating function of one-dimensional SAWs restricted to x_+ direction, while in $G_1(x_+,x_-)$ both x_+ and x_- directions are permitted. The $G_3(x_+|u_+,v_-)$ is a three-choice DSAW permitted to walk in the u_+, v_- and x_+ directions in the triangular lattice, and $G_4(x_+,x_-|u_+,v_-)$ is a four-choice DSAW restricted in x_+,x_- and u_+,v_- directions [3]. In the expressions, the bars are used to indicate symmetry relations. For an example, $G_4(x_+,x_-|u_+,v_-)$ is symmetric in x_+ and x_-, and u_+ and v_-.

As shown in figure 2, we fix the first step in the x_+ direction. It is easily seen that $\Phi_{\max}=\pi$ in the triangular lattice. Let us suppose that the walker takes a u_+ or v_- direction after the initial x_+ steps. After these u_+ or v_- steps, if the walker never takes a v_+ or u_- step, the walk becomes a four-choice DSAW permitted to take steps in x_+, x_- and v_-,u_+ directions. These walks are represented by $x_+G_1(x_+)(u_++v_-)\times G_4(x_+,x_-|u_+,v_-)$.

On the other hand, if the walker is to take any step in the v_+ direction, all walks just before the v_+ step must end with an x_+ or u_+ direction. Such walks can be written as $\{v_-G_1(v_-)(x_++u_+)+u_+\}$. After these steps, the walks would be a four-choice DSAW in v_-,v_+ and u_+,x_+ directions minus all walks not having any step in v_+ direction.

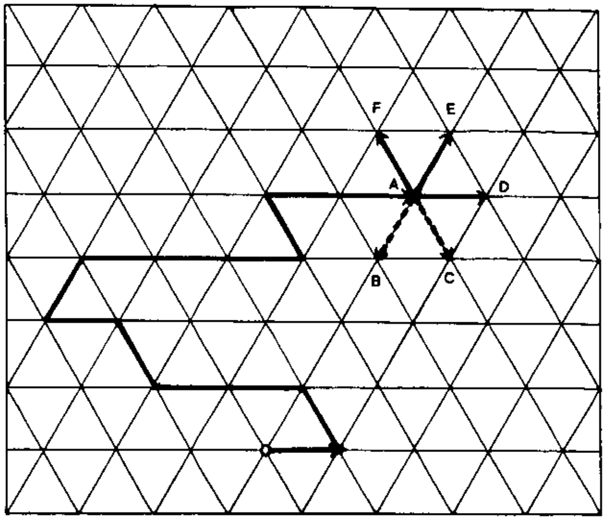

Figure 2. Walks in the triangular lattice. The walks AB, AC are prohibited, while the walks AD, AE, AF are permissible.

Such walks can be represented by

$$x_+G_1(x_+)\{v_-G_1(v_-)(x_++u_+)+u_+\}\{G_4(v_-,v_+|u_+,x_+)-G_3(v_-|u_+,x_+)\}.$$

If the walker is to take any step in u_- direction, the walks is to be represented by

$$x_+G_1(x_+)u_+G_1(u_+)(x_++v_+)\{G_4(u_+,u_-|x_+,v_+)-G_3(u_+|x_+,v_+)\}.$$

From the symmetry consideration of the model, we get similar forms for the walks turned to the v_+ or u_- direction after the initial x_+ steps by simply changing u_+ to v_+ and v_- to u_-. Therefore, the generating function for the triangular lattice is given by

$$G_{\mathrm{tri}}=\{G_1(x_+)-1\}+g(x_+:u_+,v_-)+g(x_+:v_+,u_-) \tag{2}$$

where

$$\begin{aligned} g(x_+:u_+,v_-)=&\,x_+G_1(x_+)(u_++v_-)G_4(x_+,x_-|u_+,v_-)\\ &+x_+G_1(x_+)\{v_-G_1(v_-)(x_++u_+)+u_+\}\\ &\times\{G_4(v_-,v_+|u_+,x_+)-G_3(v_-|u_+,x_+)\}\\ &+x_+G_1(x_+)u_+G_1(u_+)(x_++v_+)\\ &\times\{G_4(u_+,u_-|x_+,v_+)-G_3(u_+|x_+,v_+)\}. \end{aligned} \tag{3}$$

Performing contour integrals over the generating function [see I], a number of the desired quantities may be obtained, such as the total number of N-step walks, the total number of steps in the given direction in the ensemble of the N-step walks, and the mean square end-to-end distance. We list the results in the appendix.

2.2. Honeycomb lattice

Due to the *wavy* nature of the line in the honeycomb lattice, two values of $\Phi_{\max}$ can be defined as shown in figure 3. We call it the A-model when $\Phi_{\max}=\pi$ and attach the

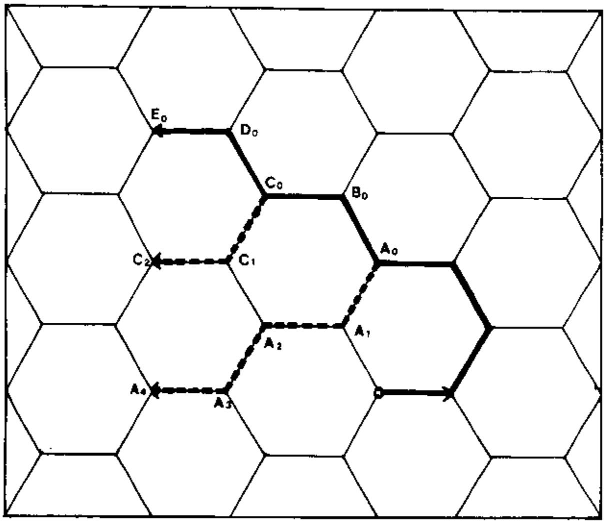

Figure 3. Two models in the honeycomb lattices. In the B-model ($\Phi_{max}=\frac{3}{2}\pi$), the walks $A_0A_1A_2A_3A_4$, and $C_0C_1C_2$ are permitted, but not in the A-model ($\Phi_{max}=\pi$). In the A-model, only walks like $A_0B_0C_0D_0E_0$ are permitted.

superscript A in all expressions for the model, while for $\Phi_{max}=\frac{3}{2}\pi$, we designate it as the B-model and use superscript B. We fix the first step to the x_+ direction, and this automatically implies that the walks cannot start from the lattice point having no x_+ branch. Therefore, care must be taken when one talks about average quantitites. However, the walks starting out in other direction can be obtained easily by the symmetry consideration, and thus any average quantities including averages over all starting points may be obtained, if desired.

For the honeycomb lattices, we need the following generating functions:

$$G_1[v_+, u_-]=\frac{v_+(1+u_-)}{1-v_+u_-} \tag{4a}$$

$$G_1[v_+, u_-, (v_+)]=\frac{v_+}{1-v_+u_-} \tag{4b}$$

$$G_1[v_+, u_-; u_+, v_-]=1+G_1[v_+, u_-]+G_1[u_+, v_-] \tag{4c}$$

$$G_1[v_+, u_-; u_+, v_-, (v_+, u_+)]=G_1[v_+, u_-, (v_+)]+G_1[u_+, v_-, (u_+)] \tag{4d}$$

$$G_2[x_+|u_+, v_-]=\frac{1+G_1[u_+, v_-]}{1-x_+G_1[u_+, v_-, (u_+)]} \tag{4e}$$

$$G_2[x_+|u_+, v_-, (v_+)]=\frac{1+G_1[u_+, v_-]+v_+}{1-x_+\{G_1[u_+, v_-, (u_+)]+v_+\}} \tag{4f}$$

$$G_3[x_+|v_+, u_-; u_+, v_-]=\frac{G_1[v_+, u_+; u_-, v_-]}{1-x_+G_1[v_+, u_-; u_+, v_-, (v_+, u_+)]}. \tag{4g}$$

Here, $G_1[v_+, u_-]$ is the generating function for a one-dimensional SAW permitted to go only one direction taking v_+ first and then u_- and v_+ steps alternately in the honeycomb lattice, while in $G_1[v_+, u_-; u_+, v_-]$ movements in both directions are permitted. Since the walker in the honeycomb needs to take two steps in order to make a turn to another direction, we need to define $G_1[v_+, u_-, (v_+)]$ etc in which the walks end up with the steps enclosed in ( ). The $G_2[x_+|v_+, v_-]$ is the generating function of a two-choice DSAW in the honeycomb lattice in which the walker is restricted to taking steps in the u_+, v_- and x_+ directions. The $G_3[x_+|v_+, u_-; u_+, v_-]$ is the generating function of a three-choice DSAW in which the walker is restricted to taking steps in (v_+, u_+) and (u_-, v_-) alternately, after x_+ steps are taken. To symbolize the *wavy* nature of the walks, we use the brackets [ ] instead of the parentheses ( ) in these expressions.

Following the similar argument as for the triangular lattice, we obtain the generating function for the A-model:

$$G^{\mathrm{A}}_{\mathrm{hc}} = x_+ + g^{\mathrm{A}}(x_+ : u_+, v_-) + g^{\mathrm{A}}(x_+ : v_+, u_-) \tag{5}$$

where

$$\begin{aligned} g^{\mathrm{A}}&(x_+ : u_+, v_-) \\ &= x_+u_+G_2[u_+|v_-, x_-, (x_+)] \\ &\quad + x_+G_1[u_+, v_-, (u_+)]x_+\{G_2[x_+|u_+, v_-, (v_+)] - G_2[x_+|u_+, v_-]\} \\ &\quad + x_+u_+G_1[x_+, u_+, (x_+)]\{G_2[v_+|x_+, u_+, (u_-)] - G_2[v_+|x_+, u_+]\}. \end{aligned} \tag{6}$$

Also for the B-model, we obtain

$$G^{\mathrm{B}}_{\mathrm{hc}} = x_+ + g^{\mathrm{B}}(x_+ : u_+, v_-) + g^{\mathrm{B}}(x_+ : v_+, u_-) \tag{7}$$

where

$$\begin{aligned} g^{\mathrm{B}}&(x_+ : u_+, v_-) \\ &= x_+G_1[u_+, x_+] \\ &\quad + x_+G_1[u_+, x_+, (u_+)]v_-G_3[v_-|x_-, u_+; u_-, x_+] \\ &\quad + x_+u_+\{G_3[u_+|v_-, x_+; x_-, v_+] - G_3[u_+|v_-, x_+; x_-]\} \\ &\quad + x_+G_1[u_+, v_-, (u_+)]x_+\{G_3[x_+|u_+, v_+; v_-, u_-)] - G_3[x_+|u_+, v_+; v_-]\} \\ &\quad + x_+u_+G_1[x_+, u_+, (x_+)]\{G_3[v_+|x_+, u_-; u_+, x_-)] \\ &\quad - G_3[v_+|x_+, u_-; u_+]\}. \end{aligned} \tag{8}$$

The $G_3[v_+|x_+, u_-; u_+]$ is the same generating function of a three-choice DSAW as $G_3[v_+|x_+, u_-; u_+, x_-]$ but the x_- step is prohibited.

We list various quantities obtained from the generating functions in the appendix.

3. Discussion

For the triangular lattice, as $N \to \infty$, the number of N-step walks, a_N, reduces to

$$a_N \sim \frac{5\sqrt{17}+17}{34}\left(\frac{\sqrt{17}+3}{2}\right)^N. \tag{9}$$

The average number of the steps in given directions are reduced to:

$$\langle x_+ \rangle \sim \left(\frac{1}{6} + \frac{17-3\sqrt{17}}{51}\right) N \tag{10a}$$

$$\langle x_- \rangle \sim \left(\frac{1}{6} - \frac{33\sqrt{17}-119}{204}\right) N \tag{10b}$$

$$\langle u_+\rangle = \langle v_+\rangle \sim \left(\frac{1}{6}+\frac{33\sqrt{17}-119}{408}\right)N \tag{10c}$$

$$\langle u_-\rangle = \langle v_-\rangle \sim \left(\frac{1}{6}-\frac{17-3\sqrt{17}}{102}\right)N. \tag{10d}$$

It is seen that the walk exhibits the characteristics reminiscent of one-dimensional SAWs in all directions. We have expressed the results in such a way that the anisotropic effects of the direction of the first step can be seen clearly.

The mean square end-to-end distance is reduced to

$$\langle R^2\rangle \sim \frac{3(9+\sqrt{17})}{136} N^2. \tag{11}$$

The results for the A-model in the honeycomb lattice show similar behaviour. For example,

$$a_N \sim \frac{5+3\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^N. \tag{12}$$

The results for the B-model in the honeycomb lattice also confirm the above conclusions. However, the model exhibits the seemingly peculiar behaviour that the even–odd oscillations in a_N, b_N and c_N do not decay as N increases. For example,

$$a_N \sim \left(\frac{12+7\sqrt{3}}{9}\right)(\sqrt{3})^N[1-(-1)^N(97-56\sqrt{3})]. \tag{13}$$

It has been known that, in the ordinary SAW, the even–odd oscillations are most significant in the honeycomb lattice, strongly favouring the odd-step walks [12–17]. This is due to the small coordination number of the honeycomb lattice, and the fact that the first intersection of the walks occurs at even steps in the ordinary SAW. The already favoured odd-step walks of the ordinary SAW are further enhanced in the B-model since the first prohibited step due to the additional restriction of the model has occurs at even steps. In the A-model, the contrary happens, and the chance of odd-step walks are suppressed. Therefore, as shown in figure 4, the even–odd oscillation in the A-model decays faster than in the ordinary SAW while the oscillation in the B-model persists forever. However, owing to its symmetry nature, the even–odd oscillation of the mean square end-to-end distance decays even in the B-model as $n\to\infty$:

$$\langle R^2\rangle \sim \tfrac{1}{4}N^2. \tag{14}$$

Acknowledgments

One of the authors (SK) would like to express his sincere thanks to Professor I Oppenheim for his hospitality at MIT and his helpful comments. IMK also would like to thank to Professor S Redner for his hospitality at Boston and the financial support by Yonam Foundation.

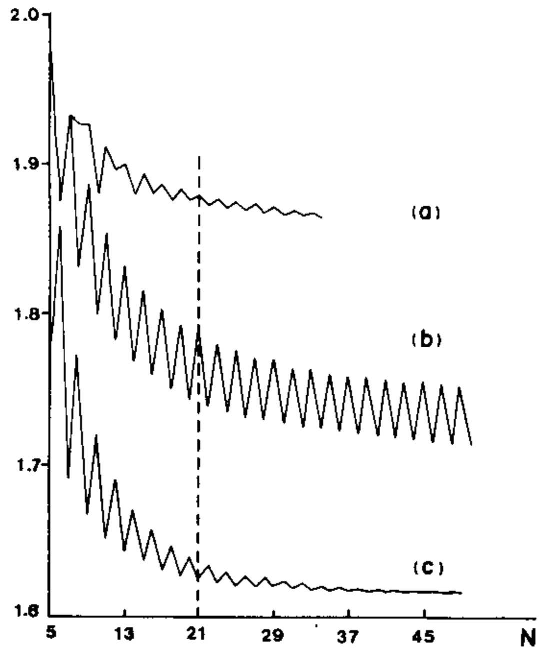

Figure 4. The behaviour of a_N in the honeycomb lattice for (a) the ordinary SAW [15], (b) the B-model and (c) the A-model. The a_N/a_{N-1} is plotted against N. As the broken vertical line at $N=21$ highlights, the already favoured odd-step walks in the ordinary SAW are enhanced in the B-model while the tendency is suppressed in the A-model.

Appendix. Exact results

In the following, we list various quantities obtained from the generating functions for the triangular and honeycomb (A- and B-model) lattices. The results are *all exact.* The a_N is the number of N step walks. The $b_N^{(x_+)}$ etc represent the total number of steps in the x_+ direction, etc, in the ensemble of the N-step walks. The $c_N^{(h)}$ ($c_N^{(v)}$) is the square of the horizontal (vertical) distance of the N-step walks. The mean square end-to-end distance is given by $\langle R^2\rangle = (c_N^{(h)}+c_N^{(v)})/a_N$.

A1. The triangular lattice

$$a_N = \left(\frac{5\sqrt{17}+17}{34}\right)\left(\frac{\sqrt{17}+3}{2}\right)^N - (-1)^N\left(\frac{5\sqrt{17}-17}{34}\right)\left(\frac{\sqrt{17}-3}{2}\right)^N - 3^N \tag{A1}$$

$$\begin{aligned} b_N^{(x_+)} = &\frac{1}{2312}\left(\frac{\sqrt{17}+3}{2}\right)^N [(102\sqrt{17}+238)N + (281\sqrt{17}+867)] \\ &- (-1)^N \frac{1}{2312}\left(\frac{\sqrt{17}-3}{2}\right)^N [(102\sqrt{17}-238)N + (281\sqrt{17}-867)] \\ &- 3^{N-1}(N+2) - \tfrac{1}{4} \end{aligned} \tag{A2}$$

$$\begin{aligned} b_N^{(x_-)} = &\frac{1}{2312}\left(\frac{\sqrt{17}+3}{2}\right)^N [(68\sqrt{17}-68)N - (289+39\sqrt{17})] \\ &- (-1)^N \frac{1}{2312}\left(\frac{\sqrt{17}-3}{2}\right)^N [(68\sqrt{17}+68)N + (289-39\sqrt{17})] + \tfrac{1}{4} \end{aligned} \tag{A3}$$

$$b_N^{(u_+)}=b_N^{(v_+)}=\frac{1}{2312}\left(\frac{\sqrt{17}+3}{2}\right)^N[(323+51\sqrt{17})N-(289+57\sqrt{17})]$$
$$+(-1)^N\frac{1}{2312}\left(\frac{\sqrt{17}-3}{2}\right)^N[(323-51\sqrt{17})N-(289-57\sqrt{17})]$$
$$-3^{N-2}2(N-1) \tag{A4}$$

$$b_N^{(u_-)}=b_N^{(v_-)}=\frac{1}{2312}\left(\frac{\sqrt{17}+3}{2}\right)^N[(170+34\sqrt{17})N-64\sqrt{17}]$$
$$+(-1)^N\frac{1}{2312}\left(\frac{\sqrt{17}-3}{2}\right)^N[(170-34\sqrt{17})N+64\sqrt{17}]$$
$$-3^{N-2}(N-1) \tag{A5}$$

$$c_N^{(h)}=\frac{1}{9248}\left(\frac{\sqrt{17}+3}{2}\right)^N\left[(765+117\sqrt{17})N^2+(987\sqrt{17}+3843)N+\left(\frac{3822\sqrt{17}+9826}{17}\right)\right]$$
$$+(-1)^N\frac{1}{9248}\left(\frac{\sqrt{17}-3}{2}\right)^N\left[(765-117\sqrt{17})N^2-(987\sqrt{17}-3843)N\right.$$
$$\left.-\left(\frac{3822\sqrt{17}-9826}{17}\right)\right]-3^{N-2}\left(\frac{4N^2+13N+1}{2}\right)-\frac{1}{8} \tag{A6}$$

$$c_N^{(v)}=\frac{3}{9248}\left(\frac{\sqrt{17}+3}{2}\right)^N[(85\sqrt{17}+221)N^2+(323-5\sqrt{17})N-(142\sqrt{17}+578)]$$
$$-(-1)^N\frac{3}{9248}\left(\frac{\sqrt{17}-3}{2}\right)^N[(85\sqrt{17}-221)N^2-(323+5\sqrt{17})N$$
$$-(142\sqrt{17}-578)]-3^{N-2}\left(\frac{(4N+1)(N-1)}{2}\right)+\frac{3}{8}. \tag{A7}$$

A2. The honeycomb lattice: A-model

$$a_N=\left(\frac{3\sqrt{5}+5}{5}\right)\left(\frac{\sqrt{5}+1}{2}\right)^N-(-1)^N\left(\frac{3\sqrt{5}-5}{5}\right)\left(\frac{\sqrt{5}-1}{2}\right)^N$$
$$-\left(\frac{3\sqrt{2}+4}{4}\right)(\sqrt{2})^N+(-1)^N\left(\frac{3\sqrt{2}-4}{4}\right)(\sqrt{2})^N \tag{A8}$$

$$b_N^{(x_+)}=\frac{1}{50}\left(\frac{\sqrt{5}+1}{2}\right)^N[(10\sqrt{5}+10)N+(23\sqrt{5}+25)-(-1)^N(25-5\sqrt{5})]$$
$$-(-1)^N\frac{1}{50}\left(\frac{\sqrt{5}-1}{2}\right)^N[(10\sqrt{5}-10)N+(23\sqrt{5}-25)+(-1)^N(25+5\sqrt{5})]$$
$$-\tfrac{1}{8}(\sqrt{2})^N[(3+2\sqrt{2})N+(4\sqrt{2}+2)]$$
$$-(-1)^N\tfrac{1}{8}(\sqrt{2})^N[(3-2\sqrt{2})N-(4\sqrt{2}-2)] \tag{A9}$$

$$b_N^{(x_-)}=\frac{1}{100}\left(\frac{\sqrt{5}+1}{2}\right)^N[(10\sqrt{5}-10)N-(25+3\sqrt{5})+(-1)^N(25-5\sqrt{5})]$$
$$-(-1)^N\frac{1}{100}\left(\frac{\sqrt{5}-1}{2}\right)^N[(10\sqrt{5}+10)N+(25-3\sqrt{5})$$
$$-(-1)^N(25+5\sqrt{5})] \tag{A10}$$

$$b_N^{(u_+)} = b_N^{(v_+)} = \frac{1}{50}\left(\frac{\sqrt{5}+1}{2}\right)^N [(15+5\sqrt{5})N - 3\sqrt{5} + (-1)^N 5\sqrt{5}]$$
$$+ (-1)^N \frac{1}{50}\left(\frac{\sqrt{5}-1}{2}\right)^N [(15-5\sqrt{5})N + 3\sqrt{5} - (-1)^N 5\sqrt{5}]$$
$$- \tfrac{1}{16}(\sqrt{2})^N [(3\sqrt{2}+4)N - 3\sqrt{2}] + (-1)^N \tfrac{1}{16}(\sqrt{2})^N$$
$$\times [(3\sqrt{2}-4)N - 3\sqrt{2}] \tag{A11}$$

$$b_N^{(u_-)} = b_N^{(v_-)} = \frac{1}{200}\left(\frac{\sqrt{5}+1}{2}\right)^N [(30+10\sqrt{5})N - (31\sqrt{5}+25) - (-1)^N(25\sqrt{5}-25)]$$
$$+ (-1)^N \frac{1}{200}\left(\frac{\sqrt{5}-1}{2}\right)^N [(30-10\sqrt{5})N + (31\sqrt{5}-25)$$
$$+ (-1)^N(25\sqrt{5}+25)] - \tfrac{1}{16}(\sqrt{2})^N [(\sqrt{2}+1)N - (2+\sqrt{2})]$$
$$+ (-1)^N \tfrac{1}{16}(\sqrt{2})^N [(\sqrt{2}-1)N + (2-\sqrt{2})] \tag{A12}$$

$$c_N^{(h)} = \frac{1}{2000}\left(\frac{\sqrt{5}+1}{2}\right)^N [(450+180\sqrt{5})N^2 + (2610+660\sqrt{5})N - (477\sqrt{5}-250)]$$
$$+ (-1)^N \frac{1}{2000}\left(\frac{\sqrt{5}-1}{2}\right)^N [(450-180\sqrt{5})N^2 + (2610-660\sqrt{5})N$$
$$+ (477\sqrt{5}+250)] - (-1)^N \frac{1}{80}\left(\frac{\sqrt{5}-1}{2}\right)^N [(6\sqrt{5})N - (27\sqrt{5}-10)]$$
$$+ (-1)^N \frac{1}{80}\left(\frac{\sqrt{5}+1}{2}\right)^N [(6\sqrt{5})N - (27\sqrt{5}+10)]$$
$$- \tfrac{1}{128}(\sqrt{2})^N [(45+27\sqrt{2})N^2 + (60\sqrt{2}+54)N + 9\sqrt{2}]$$
$$- (-1)^N \tfrac{1}{128}(\sqrt{2})^N [(45-27\sqrt{2})N^2 - (60\sqrt{2}-54)N - 9\sqrt{2}] \tag{A13}$$

$$c_N^{(v)} = \frac{3}{2000}\left(\frac{\sqrt{5}+1}{2}\right)^N [(80\sqrt{5}+150)N^2 + (20\sqrt{5}+10)N - (500+7\sqrt{5})]$$
$$- (-1)^N \frac{3}{2000}\left(\frac{\sqrt{5}-1}{2}\right)^N [(80\sqrt{5}-150)N^2 + (20\sqrt{5}-10)N + (500-7\sqrt{5})]$$
$$+ (-1)^N \frac{3}{80}\left(\frac{\sqrt{5}-1}{2}\right)^N [(2\sqrt{5})N + (20+\sqrt{5})]$$
$$- (-1)^N \frac{3}{80}\left(\frac{\sqrt{5}+1}{2}\right)^N [(2\sqrt{5})N - (20-\sqrt{5})]$$
$$- \frac{3}{128}(\sqrt{2})^N [(9\sqrt{2}+9)N^2 - (12\sqrt{2}+2)N + 3\sqrt{2}]$$
$$+ (-1)^N \tfrac{3}{128}(\sqrt{2})^N [(9\sqrt{2}-9)N^2 - (12\sqrt{2}-2)N + 3\sqrt{2}]. \tag{A14}$$

A3. The honeycomb lattice: B-model

$$a_N=\left(\frac{7\sqrt{3}+12}{9}\right)(\sqrt{3})^N-(-1)^N\left(\frac{7\sqrt{3}-12}{9}\right)(\sqrt{3})^N$$
$$-\left(\frac{3\sqrt{5}+5}{5}\right)\left(\frac{\sqrt{5}+1}{2}\right)^N+(-1)^N\left(\frac{3\sqrt{5}-5}{5}\right)\left(\frac{\sqrt{5}-1}{2}\right)^N \tag{A15}$$

$$b_N^{(x_+)}=\tfrac{1}{108}(\sqrt{3})^N[(33+19\sqrt{3})N+(78\sqrt{3}+105)]$$
$$+(-1)^N\tfrac{1}{108}(\sqrt{3})^N[(33-19\sqrt{3})N-(78\sqrt{3}-105)]$$
$$-\frac{1}{100}\left(\frac{\sqrt{5}+1}{2}\right)^N[(20\sqrt{5}+20)N+(46\sqrt{5}+50)-(-1)^N(50-10\sqrt{5})]$$
$$+(-1)^N\frac{1}{100}\left(\frac{\sqrt{5}-1}{2}\right)^N[(20\sqrt{5}-20)N+(46\sqrt{5}-50)$$
$$+(-1)^N(50+10\sqrt{5})]-\tfrac{1}{2}(-1)^N-1 \tag{A16}$$

$$b_N^{(x_-)}=\tfrac{1}{36}(\sqrt{3})^N[(3\sqrt{3}+5)N-(12\sqrt{3}+13)]$$
$$-(-1)^N\tfrac{1}{36}(\sqrt{3})^N[(3\sqrt{3}-5)N-(12\sqrt{3}-13)]$$
$$-\frac{1}{100}\left(\frac{\sqrt{5}+1}{2}\right)^N[(10\sqrt{5}-10)N-(25+3\sqrt{5})+(-1)^N(25-5\sqrt{5})]$$
$$+(-1)^N\frac{1}{100}\left(\frac{\sqrt{5}-1}{2}\right)^N[(10\sqrt{5}+10)N+(25-3\sqrt{5})-(-1)^N(25+5\sqrt{5})]$$
$$-\tfrac{1}{2}(-1)^N+1 \tag{A17}$$

$$b_N^{(u_+)}=b_N^{(v_+)}=\tfrac{1}{72}(\sqrt{3})^N[(11\sqrt{3}+19)N+(13-2\sqrt{3})]$$
$$-(-1)^N\tfrac{1}{72}(\sqrt{3})^N[(11\sqrt{3}-19)N-(13+2\sqrt{3})]$$
$$-\frac{1}{200}\left(\frac{\sqrt{5}+1}{2}\right)^N[(60+20\sqrt{5})N-12\sqrt{5}+(-1)^N(20\sqrt{5})]$$
$$-(-1)^N\frac{1}{200}\left(\frac{\sqrt{5}-1}{2}\right)^N[(60-20\sqrt{5})N+12\sqrt{5}-(-1)^N(20\sqrt{5})]$$
$$+\tfrac{1}{4}(-1)^N-\tfrac{1}{2} \tag{A18}$$

$$b_N^{(u_-)}=b_N^{(v_-)}=\tfrac{1}{216}(\sqrt{3})^N[(23\sqrt{3}+39)N-(105+36\sqrt{3})]$$
$$-(-1)^N\tfrac{1}{216}(\sqrt{3})^N[(23\sqrt{3}-39)N+(105-36\sqrt{3})]$$
$$-\frac{1}{200}\left(\frac{\sqrt{5}+1}{2}\right)^N[(30+10\sqrt{5})N-(31\sqrt{5}+25)-(-1)^N(25\sqrt{5}-25)]$$
$$-(-1)^N\frac{1}{200}\left(\frac{\sqrt{5}-1}{2}\right)^N[(30-10\sqrt{5})N+(31\sqrt{5}-25)$$
$$+(-1)^N(25\sqrt{5}+25)]+\tfrac{1}{4}(-1)^N+\tfrac{1}{2} \tag{A19}$$

$$c_N^{(h)}=\tfrac{1}{144}(\sqrt{3})^N[(13\sqrt{3}+21)N^2+(234+116\sqrt{3})N-(174+23\sqrt{3})]$$
$$-(-1)^N\tfrac{1}{144}(\sqrt{3})^N[(13\sqrt{3}-21)N^2-(234-116\sqrt{3})N+(174-23\sqrt{3})]$$
$$-\frac{1}{2000}\left(\frac{\sqrt{5}+1}{2}\right)^N[(450+180\sqrt{5})N^2+(2610+660\sqrt{5})N-(477\sqrt{5}-250)]$$
$$-(-1)^N\frac{1}{2000}\left(\frac{\sqrt{5}-1}{2}\right)^N[(450-180\sqrt{5})N^2+(2610-660\sqrt{5})N$$
$$+(477\sqrt{5}+250)]-(-1)^N\frac{1}{80}\left(\frac{\sqrt{5}+1}{2}\right)^N[(6\sqrt{5})N-(27\sqrt{5}+10)]$$
$$+\frac{1}{80}\left(\frac{\sqrt{5}-1}{2}\right)^N[(6\sqrt{5})N-(27\sqrt{5}-10)]+(-1)^N\tfrac{3}{4}+\tfrac{3}{2} \qquad \text{(A20)}$$

$$c_N^{(v)}=\tfrac{1}{144}(\sqrt{3})^N[(27+15\sqrt{3})N^2+(76\sqrt{3}+102)N-(213\sqrt{3}+258)]$$
$$+(-1)^N\tfrac{1}{144}(\sqrt{3})^N[(27-15\sqrt{3})N^2-(76\sqrt{3}-102)N+(213\sqrt{3}-258)]$$
$$-\frac{1}{2000}\left(\frac{\sqrt{5}+1}{2}\right)^N[(240\sqrt{5}+450)N^2+(60\sqrt{5}+30)N-(1500+21\sqrt{5})]$$
$$+(-1)^N\frac{1}{2000}\left(\frac{\sqrt{5}-1}{2}\right)^N[(240\sqrt{5}-450)N^2+(60\sqrt{5}-30)N$$
$$+(1500-21\sqrt{5})]+(-1)^N\frac{1}{80}\left(\frac{\sqrt{5}+1}{2}\right)^N[(6\sqrt{5})N-(60-3\sqrt{5})]$$
$$-\frac{1}{80}\left(\frac{\sqrt{5}-1}{2}\right)^N[(6\sqrt{5})N+(60+3\sqrt{5})]-(-1)^N\tfrac{3}{4}+\tfrac{9}{2}. \qquad \text{(A21)}$$

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